Journal of Approximation Theory **93**, 450–457 (1998)
Article No. AT983167

A Remez-Type Inequality for Non-dense Müntz Spaces with Explicit Bound*

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Communicated by András Kroó

Received June 12, 1996; accepted in revised form May 1, 1997

Let $\Lambda := (\lambda_k)_{k=0}^{\infty}$ be a sequence of distinct nonnegative real numbers with $\lambda_0 := 0$ and $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$. Let $\varrho \in (0, 1)$ and $\varepsilon \in (0, 1-\varrho)$ be fixed. An earlier work of the authors shows that

$$C(\Lambda, \varepsilon, \varrho) := \sup\{ \|p\|_{[0, \varrho]} : p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, ...\}, \quad m(\{x \in [\varrho, 1] : |p(x)| \leqslant 1\}) \geqslant \varepsilon\}$$

is finite. In this paper an explicit upper bound for $C(\Lambda, \varepsilon, \varrho)$ is given. In the special case $\lambda_k := k^{\alpha}$, $\alpha > 1$, our bounds are essentially sharp.

1. INTRODUCTION

In this paper $\Lambda := (\lambda_k)_{k=0}^{\infty}$ always denotes a sequence of real numbers satisfying

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots.$$

* Research of the first author supported, in part, by NSERC of Canada. Research of the second author supported, in part, by NSF under Grant DMS-9024901 and conducted while an NSERC International Fellow at Simon Fraser University.

0021-9045/98 $25.00

In [1] a Remez-type inequality for Müntz polynomials

$$p(x) = \sum_{k=0}^{n} a_k x^{\lambda_k}$$

or equivalently for Dirichlet sums

$$P(t) = \sum_{k=0}^{n} a_k e^{-\lambda_k t}$$

is established. The most useful form of this inequality states that for every sequence $(\lambda_k)_{k=0}^{\infty}$ satisfying $\sum_{k=0}^{\infty} 1/\lambda_k < \infty$, there exists a constant $C(\Lambda, \varepsilon)$ depending only on Λ and ε (and not on n, ϱ, or A) so that

$$\|p\|_{[0,\varrho]} \leqslant C(\Lambda, \varepsilon)\, \|p\|_A$$

for every Müntz polynomial p, as above, associated with the sequence $(\lambda_k)_{k=0}^{\infty}$, and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least $\varepsilon > 0$. Throughout this paper $\|\cdot\|_A$ denotes the uniform norm on $A \subset \mathbb{R}$.

Using this Remez-type inequality, we resolved two reasonably long standing conjectures in [1]. In this paper we give an explicit upper bound for the best possible $C(\Lambda, \varepsilon)$ in the above Remez-type inequality for non-dense Müntz spaces. Theorem 2.3 extends an inequality of Schwartz [4] in two directions. Theorem 2.1 offers a more explicit bound for the sequences $\Lambda := (k^{\alpha})_{k=0}^{\infty}$, $\alpha > 1$. The sharpness of the Remez-type inequality of Theorem 2.1 is shown by Theorem 2.2.

2. RESULTS

THEOREM 2.1. *Let* $\lambda_k := k^{\alpha}$, $k = 0, 1, \ldots$, $\alpha > 1$. *Let* $\varrho \in (0, 1)$, $\varepsilon \in (0, 1-\varrho)$, *and* $\varepsilon \leqslant 1/2$. *There exists a constant* $c_\alpha > 0$ *depending only on* α *so that*

$$\|p\|_{[0,\varrho]} \leqslant \exp(c_\alpha \varepsilon^{1/(1-\alpha)})\, \|p\|_A$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$ *and for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* $\varepsilon > 0$.

The next theorem shows that the inequality of Theorem 2.1 is essentially the best possible.

THEOREM 2.2. *Let* $\lambda_k := k^{\alpha}$, $k = 0, 1, \ldots$, $\alpha > 1$. *For every* $\alpha > 1$ *and* $\varepsilon \in (0, 1/2]$, *there exists a constant* $c_\alpha > 0$ *depending only on* α *and Müntz polynomials*

$$0 \neq p = p_{\alpha,\varepsilon} \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$$

depending only on α *and* ε *so that*

$$|p(0)| \geqslant \exp(c_\alpha \varepsilon^{1/(1-\alpha)}) \, \|p\|_{[1-\varepsilon]}.$$

Theorem 2.1 is a special case of the following more general, but less explicit result.

THEOREM 2.3. *Suppose* $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ *and* $\sum_{k=0}^{\infty} 1/\lambda_k < \infty$. *Let* $\varrho \in (0, 1)$ *and* $\varepsilon \in (0, 1-\varrho)$. *Let* $\delta := -\frac{1}{2}\log(1-\varepsilon)$. *Let* $N \in \mathbb{N}$ *be chosen so that*

$$\sum_{k=N+1}^{\infty} \frac{1}{\lambda_k} \leqslant \frac{\delta}{3}.$$

Let

$$\sigma_k := A\lambda_k \qquad \textit{with} \quad A := \frac{\delta}{3N}.$$

Then, with $c := \|t^{-1} \sin t\|_{L_2(\mathbb{R})}$,

$$\|p\|_{[0,\varrho]} \leqslant \frac{3c}{\delta} \prod_{k=1}^{N} \left(2 + \frac{1}{\sigma_k}\right) \|p\|_A$$

for every $p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, ...\}$ *and for every set* $A \subset [\varrho, 1]$ *of Lebesque measure at least* $\varepsilon > 0$.

3. LEMMAS

Our first lemma shows that $C(\Lambda, \varepsilon)$ in the Remez-type inequality is related to a much simpler (Chebyshev-type) extremal problem. This is proved in [1, 2].

LEMMA 3.1. *Suppose* $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, $\rho \in (0, 1)$, *and* $\varepsilon \in (0, 1-\rho)$. *Then*

$$\sup\{\|p\|_{[0,\varrho]} : p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, ...\}, m\{x \in [\varrho, 1] : |p(x)| \leqslant 1\} \geqslant \varepsilon\}$$
$$= \sup\left\{\frac{|p(0)|}{\|p\|_{[1-\varepsilon,1]}} : p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, ...\}\right\}.$$

Our key lemma is the following.

Lemma 3.2. *Suppose* $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ *and* $\sum_{k=0}^{\infty} 1/\lambda_k < \infty$. *Given* $\delta \in (0, 1)$, *let* $N \in \mathbb{N}$ *be chosen so that*

$$\sum_{k=N+1}^{\infty} \frac{1}{\lambda_k} \leqslant \frac{\delta}{12}.$$

Let

$$\sigma_k := A\lambda_k \qquad \text{with} \quad A := \frac{\delta}{3N}.$$

Then

$$|P(\infty)| \leqslant \frac{3c}{\delta} \prod_{k=1}^{N} \left(2 + \frac{1}{\sigma_k}\right) \|P\|_{[-\delta, \delta]}$$

for every $P \in \operatorname{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, ...\}$ *with* $c := \|t^{-1} \sin t\|_{L_2(\mathbb{R})}$.

In the proof of Lemma 3.2 we will need the following observation.

Lemma 3.3. *Let* $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. *Suppose*

(1) $F \in E^{\delta} \cap L_2(\mathbb{R})$;

(2) $F(i\lambda_k) = 0$, $k = 1, 2, ...$ (*i is the imaginary unit*);

(3) $F(0) = 1$.

Then

$$|P(\infty)| \leqslant \|F\|_{L_2(\mathbb{R})} \|P\|_{L_2[-\delta, \delta]}$$

for every $P \in \operatorname{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, ...\}$.

An entire function f is called a function of *exponential type* δ if there exists a constant c depending only on f so that

$$|f(z)| \leqslant c \exp(\delta\, |z|), \qquad z \in \mathbb{C}.$$

The collection of all such entire functions of exponential type δ is denoted be E^{δ}. The Paley–Wiener Theorem (see, for example, [3]) characterizes the functions F which can be written as the Fourier transform of some function $f \in L_2[-\delta, \delta]$. We will need it in the proof of Lemma 3.3.

Theorem (Paley–Wiener). *Let* $\delta \in (0, \infty)$. *Then* $f \in E^{\delta} \cap L_2(\mathbb{R})$ *if and only if there exists an* $f \in L_2[-\delta, \delta]$ *so that*

$$F(z) = \int_{-\delta}^{\delta} f(t)\, e^{itz}\, dz.$$

The following comparison theorem for Müntz polynomials is proved in [2]. We will need it in the proof of Theorem 2.3.

LEMMA 3.4. *Let* $\Lambda := (\lambda_k)_{k=0}^{\infty}$ *and* $\Gamma := (\gamma_k)_{k=0}^{\infty}$ *be increasing sequences of nonnegative real numbers with* $\lambda_0 = 0$, $\gamma_0 = 0$, *and* $\lambda_k \leqslant \gamma_k$ *for each* k. *Let* $0 < a < b$. *Then*

$$\max\left\{\frac{|p(0)|}{\|p\|_{[a,b]}} : p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, ..., x^{\lambda_n}\}\right\}$$
$$\geqslant \max\left\{\frac{|p(0)|}{\|p\|_{[a,b]}} : p \in \operatorname{span}\{x^{\gamma_0}, x^{\gamma_1}, ..., x^{\gamma_n}\}\right\}.$$

4. PROOFS

Proof of Lemma 3.3. By the Paley–Wiener Theorem

$$F(z) = \int_{-\delta}^{\delta} f(t)\, e^{itz}\, dt$$

for some $f \in L_2[-\delta, \delta]$. Now if

$$P(t) = a_0 + \sum_{k=1}^{n} a_k e^{-\lambda_k t},$$

then

$$\int_{-\delta}^{\delta} f(t)\, P(t)\, dt = a_0 \int_{-\delta}^{\delta} f(t)\, dt + \sum_{k=1}^{n} a_k \int_{-\delta}^{\delta} f(t)\, e^{-\lambda_k t}\, dt$$
$$= a_0 F(0) + \sum_{k=1}^{n} a_k F(i\lambda_k) = a_0 = P(\infty).$$

Hence by the Cauchy–Schwartz Inequality and the L_2 inversion theorem of Fourier transforms, we obtain

$$|P(\infty)| \leqslant \|f\|_{L_2[-\delta,\delta]}\, \|P\|_{L_2[-\delta,\delta]} \leqslant \|F\|_{L_2(\mathbb{R})}\, \|P\|_{L_2[-\delta,\delta]}$$

and the lemma is proved. ∎

Proof of Lemma 3.2. We define

$$F(z) := \frac{\sin(\delta z/3)}{\delta z/3} \prod_{k=1}^{N} \left(\left(1 - \frac{z}{i\lambda_k}\right) \frac{\sin(\sigma_k z/\lambda_k)}{\sigma_k z/\lambda_k}\right) \prod_{k=N+1}^{\infty} \left(1 - \left(\frac{\sin(z/\lambda_k)}{\sin i}\right)^4\right),$$

where i is the imaginary unit. It is a straightforward calculation that

$$F \in E^{\delta}, \qquad F(0)=1, \qquad F(i\lambda_k)=0, \qquad k=1, 2, \ldots$$

and

$$|F(t)| \leqslant \frac{\sin(\delta t/3)}{\delta t/3} \prod_{k=1}^{N} \left(2+\frac{1}{\sigma_k}\right), \qquad t \in \mathbb{R}.$$

Hence Lemma 3.3 implies that

$$|P(\infty)| \leqslant \frac{3c}{\delta} \prod_{k=1}^{N} \left(2+\frac{1}{\sigma_k}\right) \|P\|_{[-\delta, \delta]}$$

for every $P \in \operatorname{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \ldots\}$ with $c := \|t^{-1} \sin t\|_{L_2(\mathbb{R})}$. ▮

Proof of Theorem 2.3. When $A=[1-\varepsilon, 1]$, the theorem follows from Lemma 3.2 by the substitution $x\colon e^{-\delta} e^{-t}$. The general case follows from Lemma 3.1. ▮

Proof of Theorem 2.1. Let

$$\delta := -\tfrac{1}{2}\log(1-\varepsilon). \tag{4.1}$$

Observe that N in Theorem 2.1 can be chosen so that

$$N := \left\lfloor \left(\frac{\delta(\alpha-1)}{12}\right)^{1/(1-\alpha)} \right\rfloor + 1. \tag{4.2}$$

Also, σ_k in Lemma 3.2 is of the form

$$\sigma_k = \frac{\delta k^{\alpha}}{3N}.$$

Let $M+1$ be the smallest value of $k \in \mathbb{N}$ for which

$$\frac{1}{\sigma_k} < 1, \qquad \text{that is,} \qquad \frac{3N}{k^{\alpha}\delta} \leqslant 1.$$

Note that

$$M := \left\lfloor \left(\frac{3N}{\delta}\right)^{1/\alpha} \right\rfloor.$$

If $0 < M < N$, then

$$\begin{aligned}\prod_{k=1}^{N}\left(2+\frac{1}{\sigma_k}\right) &= \prod_{k=1}^{N}\left(2+\frac{3N}{\delta k^\alpha}\right)\\ &\leqslant \left(\prod_{k=1}^{M}\frac{9N}{\delta k^\alpha}\right)\left(\prod_{k=M+1}^{N}3\right) \leqslant \left(\frac{9N}{\delta}\right)^M\left(\frac{M}{\varepsilon}\right)^{-\alpha M} 3^{N-M}\\ &= \left(\frac{9e^\alpha N}{\delta}\right)^M M^{-\alpha M} 3^{N-M}\\ &\leqslant \left(\frac{9e^\alpha N}{\delta}\right)^M \left(\frac{1}{2}\left(\frac{3N}{\delta}\right)^{1/\alpha}\right)^{-\alpha M} 3^{N-M}\\ &\leqslant (3(2e)^\alpha)^M\, 3^{N-M} \leqslant (3(2e)^\alpha)^N,\end{aligned}$$

and the theorem follows by (4.1),(4.2), and Theorem 2.1.

If $N \leqslant M$, then

$$\begin{aligned}\prod_{k=1}^{N}\left(2+\frac{1}{\sigma_k}\right) &= \prod_{k=1}^{N}\left(2+\frac{3N}{\delta k^\alpha}\right)\\ &\leqslant \left(\prod_{k=1}^{N}\frac{9N}{\delta k^\alpha}\right) \leqslant \left(\frac{9N}{\delta}\right)^N\left(\frac{N}{e}\right)^{-\alpha N}\\ &= \left(\frac{9e^\alpha N^{1-\alpha}}{\delta}\right)^N \leqslant \left(\frac{9e^\alpha}{\delta}\right)^N \left(\left(\frac{\delta(\alpha-1)}{12}\right)^{1/(1-\alpha)}\right)^{(1-\alpha)\,N}\\ &\leqslant \left(\frac{9e^\alpha}{\delta}\right)^N\left(\frac{\delta(\alpha-1)}{12}\right)^N \leqslant \left(\frac{3e^\alpha(\alpha-1)}{4}\right)^N,\end{aligned}$$

and the theorem follows by (4.1), (4.2), and Theorem 2.1.

If $M = 0$, then

$$\prod_{k=1}^{N}\left(2+\frac{1}{\sigma_k}\right) \leqslant \prod_{k=1}^{N} 3 = 3^N,$$

and the theorem follows by (4.1), (4.2), and Theorem 2.1. ∎

Proof of Theorem 2.2. Let $n \in \mathbb{N}$ be fixed. We define $\gamma_k := kn^{\alpha-1}$, $k = 0, 1, \ldots$. Let $T_n(x) := (\frac{1}{2}(x-1))^n$ and

$$Q_n(x) := T_n\left(\frac{2x^{n^{\alpha-1}}}{1-(1-\varepsilon)^{n^{\alpha-1}}} - \frac{1+(1-\varepsilon)^{n^{\alpha-1}}}{1-(1-\varepsilon)^{n^{\alpha-1}}}\right)^n \in \operatorname{span}\{x^{\gamma_0}, x^{\gamma_1}, \ldots, x^{\gamma_n}\}.$$

Then, by Lemma 3.4,

$$\sup \left\{ \frac{|p(0)|}{\|p\|_{[1-\varepsilon, 1]}} : p \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, ...\} \right\} \leqslant \frac{|Q_n(0)|}{\|Q_n\|_{[1-\varepsilon, 1]}} = |Q_n(0)|$$

$$= \left(\frac{1}{1-(1-\varepsilon)^{n^{\alpha-1}}} \right)^n.$$

Now let n be the smallest integer satisfying $n^{\alpha-1} \geqslant \varepsilon^{-1}$. Since $(1-\varepsilon)^{1/\varepsilon}$ is bounded away form 0 on $(0, 1/2]$, the result follows. ∎

ACKNOWLEDGMENTS

The authors thank Gábor Halász for suggesting Theorem 2.3 to establish Remez-type inequalities for non-dense Müntz spaces with good explicit bounds. We also thank him for helping us to close the gap between the upper bound of Theorem 2.1 and the lower bound of Theorem 2.2.

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